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PERFORMANCE EQUATIONS FOR AN OPTIMIZED MULTIPING DATA PROCESSOR--ETC(U)

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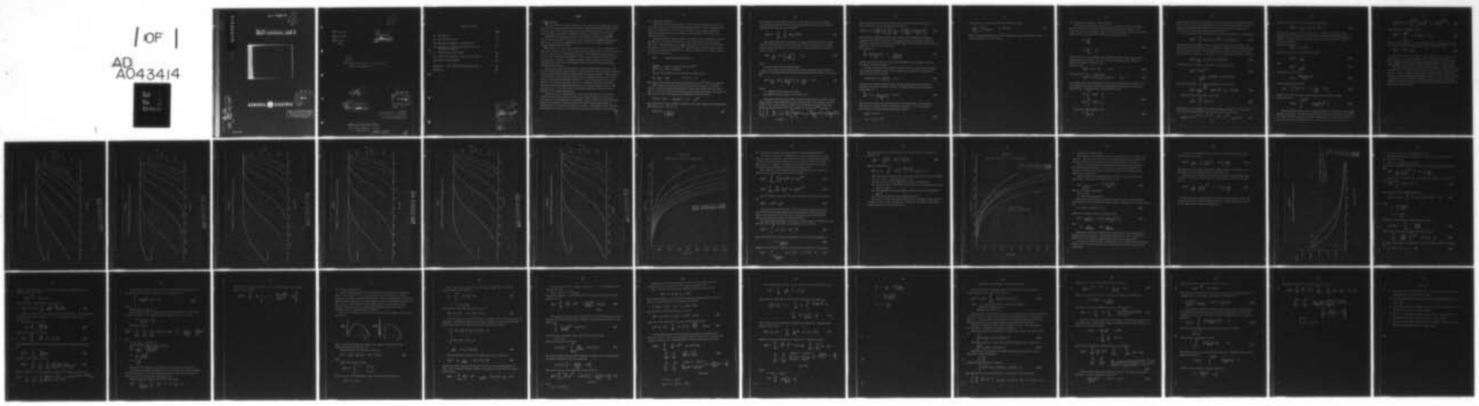

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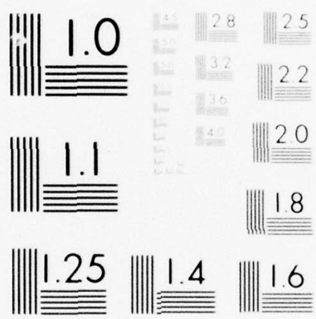

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I. INTRODUCTION

The continuing need for improved detection performance by surface ship sonars has led to the proposed development of large, high-powered Conformal/Planar Array sonar systems. These systems enhance detection through increased surveillance coverage, higher resolution capability and higher transmitted power levels.

Much of the potential enhancement of detection is due to the increased volume of data inherent to the system. Thus parallel to the physical development of an array, means must be devised to process and extract target information from very large volumes of data.

Two distinct aspects of data processing are evident. First, reduction of data volume within a single sonar ping interval and second, the intraping processing of data collected over several ping intervals. The multiping processing may be considered as the digital automation of the detection function normally carried out by a sonar operator. It involves a simple track and implies a degree of classification but is to be distinguished from the accurate track and detailed classification functions.

The current report is the third in a series on multi-ping processing. The first (1) considered an M/N processor; one which required M geometrically correlated single ping threshold crossings within a span of N ping intervals. The performance of this processor was below expectations and was adjudged inadequate due to the poor use of available amplitude information. In the second report (2), a formulation of an optimum multi-ping detector is undertaken which closely follows well developed radar design techniques for the processing of pulse trains. The formulation takes into account the stationarity differences between the sonar problem and the ideal radar problem.

The intent of the current report was to extend the work of the second to include a quantitative performance evaluation using current best estimated of the statistical distribution of sonar signals. Signal models and their associated performance equations were developed but the effort was terminated prior to programming of the equations on a digital computer. This report records the work accomplished and serves as a possible starting point for future investigations.

II. THE OPTIMUM PROCESSOR

A detailed derivation of the form of the theoretical optimum data processor for a sonar system was derived in (2). Portions of that work are repeated here and may be recognized as the classical development for testing statistical hypotheses given N samples from a universe with an assumed distribution but unknown parameters.

Let D be the total number of discrete locations under surveillance. Let A be the number of locations accessible to the target between successive ping intervals. Then the number of possible target paths after two intervals is DA. Following the N^{th} ping, there are DA^{N-1} possible paths which the target may have traveled through the surveillance area. The data processor is to decide between the alternate hypotheses:

$H(1)$: target is present with path 1

-

-

-

$H(DA^{N-1})$: target is present with path DA^{N-1}

$H(DA^{N-1}+1)$: target is absent.

The data from ping k may be summarized in vector form as

$$Z_k = (R_{k1}, \ldots R_{kN}) \qquad k = 1, 2, \ldots N$$

where R_{ki} is the single ping output corresponding to the i^{th} location on ping k.

It is desired to process the data from successive pings in a way which maximizes the probability of detection for a specified average number of false alarms. It is well known that the data processor which accomplishes this objective must compute the set of a posteriori probability density functions

$$L(Z,N,j) = p(Z_1, \ldots Z_N \mid H(j)) \quad 1.C.=1, \ldots DA^{N-1}$$

and compare these with a threshold C which is inversely weighted by the probability that $H(j)$ is true. That is, if

$$L(Z,N,j) > \frac{C}{P_r\left\{H(j)\right\}} \tag{1}$$

accept $H(j)$, otherwise $H(j)$ is rejected. The single ping processor outputs corresponding to different locations and/or different pings are assumed to be independent allowing the representation

$$L(Z,N,j) = \prod_{k=1}^{N} \prod_{i=1}^{D} p(R_{ki} \mid H(j)) \tag{2}$$

The single ping processor consists of a bank of D matched filter/envelope detector combinations, one for each location in the surveillance area. With target absent, the output is assumed to have a Rayleigh distribution given in the usual form as

$$p(R) = \frac{R}{\psi_o} \exp\left[-\frac{R^2}{2\psi_o}\right] \qquad R \geq 0 \tag{3}$$

where

R = amplitude of the voltage envelope

ψ_o = mean square value of noise voltage. With target present, the output is assumed to have a Rice distribution (non-scintillating target model) and is given by

$$P(R) = \frac{R}{\psi_o} \exp\left[-\frac{R^2 + P^2}{2\psi_o}\right] I_o\left(\frac{RP}{\psi_o}\right) \qquad R \geq 0 \tag{4}$$

where

P = amplitude of the sine wave signal

I_o = modified Bessel function of the first kind

The optimum processor must compute the expression given by equation 2 where $p(R_{ki} \mid H(j))$ is Rice distributed if the indices k and i correspond to the true target location under hypothesis $H(j)$, and is Rayleigh distributed otherwise. Substituting equations 3 and 4 into 2 yields for the first DA^{N-1} hypotheses

$$\prod_{k=1}^{N} \left\{ \prod_{\substack{i=1 \\ i \neq a(kj)}}^{D} \frac{R_{ki}}{\psi_o} \exp\left[-\frac{R_{ki}}{2\ {}_o}\right] \right\} \frac{R_{ka(kj)}}{\psi_o} \exp\left[-\frac{R^2_{ka(kj)} + P^2}{2\psi_o}\right] I_o\left(\frac{R_{ka(kj)}P}{2\psi_o}\right) \tag{5}$$

where $a(k,j)$ is the true location of the target on ping k under hypothesis j. The test function may be rewritten more simply as

$$L(R,N,j) = \exp\left[-\frac{NP^2}{2\psi_o}\right] \prod_{k=1}^{N} \left\{ \prod_{i=1}^{D} \frac{R_{ki}}{\psi_o} \exp\left[-\frac{R_{ki}^2}{2\psi_o}\right] I_o\left(\frac{R_{ka(kj)}\,P}{\psi_o}\right)\right\} \tag{6}$$

Those factors of $L(R,N,j)$ which are the same for each hypothesis need not be computed since they will have no effect on the relative magnitude of $L(R,N,j)$. Similarly, the constant factor $\exp\left[-NP^2/2\psi_o\right]$ may be included in the comparison circuitry. Thus the test reduces to:

Accept $H(j)$ if

$$\prod_{k=1}^{N} I_o\left(\frac{R_{ka(kj)}\,P}{\psi_o}\right) > \frac{C}{P_r\left\{H(m)\right\}} \tag{7}$$

If the test function does not exceed the threshold for any possible path, then the target is declared absent.

Now that the optimal test function has been obtained, any monotonic function of it may be used just as well. It is convenient to consider the logarithm of $L(R,N,j)$ which yields the form

$$\ln L(R,N,j) + \ln P_r\left\{H(j)\right\} > \ln C \tag{8}$$

If we assume all paths equally likely, then the path probability contributes equally to all tests and may be ignored. The final decision criteria becomes:

Accept $H(j)$ if

$$\sum_{k=1}^{N} \ln I_o\left(\frac{R_{k,a(kj)}\,P}{\psi_o}\right) > \ln C \tag{9}$$

Thus the optimum processor takes the sum of N variates where each variate has been processed by the function indicated in equation 9. For small signal to noise ratio, the following expansions may be applied

$$I_o(X) = 1 + \tfrac{1}{4}X^2$$

$$\ln\left(1 + \tfrac{1}{4}X^2\right) = \tfrac{1}{4}X^2 \tag{10}$$

Inserting these expansions in the test yields the result

$$\frac{P^2}{16\psi_o^2} \sum R^2_{k,a(kj)} > \text{constant} \tag{11}$$

that is, the optimal processor can be replaced by a square law detector with a linear integrator for small signals.

III PERFORMANCE EVALUATION - STEADY SIGNALS

The optimum processor has been shown to consist of the linear integration of some function of the envelope of the single ping processor output. It is convenient to take the output as that of a square law detector and normalize with respect to the mean square noise voltage toward this end, let

$$y = \frac{R^2}{2\psi_o}$$

$$x = \frac{P^2}{2\psi_o} = \frac{S}{N} \tag{12}$$

The quantity x may be identified with the signal-to-noise power ratio commonly used in detection analyses. In terms of the new variables, the single ping output distribution of interest are:

For noise alone

$$f_o(y) = \exp\left[-y\right] \qquad y > 0 \tag{13}$$

For signal plus noise -- steady signal

$$f_1(y) = \exp\left[-y-x\right] I_o\left(2\sqrt{xy}\right) \qquad y > 0 \tag{14}$$

For a single ping, the probability of false alarm and detection are equal to the probability of exceeding a preset threshold with the target absent and present respectively. In equation form, the probabilities are:

$$P_A = \int_c^\infty f_o(y)\, d_y$$

$$P_D = \int_c^\infty f_1(y)\, d_y \tag{15}$$

For the sum of N variables, let

$$Y = \sum_{i=1}^{N} y_i \tag{16}$$

The distribution functions of Equation 15 are replaced by the distribution function for Y. This is most easily obtained through the use of characteristic functions wherein the characteristic function of the sum of N variates is equal to the product of their individual characteristic functions.

For signal plus noise, the characteristic function of Equation 14 is given by

$$C_1(p) = \int_0^{\infty} \exp\left[-x-y\right] I_o\left(2\sqrt{xy}\right) \exp\left[py\right] dy \tag{17}$$

The integral may be obtained from pair 655.1 of Campbell and Foster [3]. The integrals of this reference use exp $\left[-py\right]$ for the first integration and exp $\left[py\right]$ for the inverse transform. This same practice is used here to allow direct use of the tables of reference 3. Inserting the -p in Equation 17 yields the function

$$C_1(p) = \frac{1}{p+1} \exp\left[-x\right] \exp\left[x/(p+1)\right] \tag{18}$$

The characteristic function for the sum of N variates is then

$$C_N(p) = \left[C_1(p)\right]^N \tag{19}$$

$$= \frac{1}{(p+1)^N} \exp\left[-Nx\right] \exp\left[Nx/(p+1)\right]$$

From pair 650.0 [3], the density distribution is seen to be

$$f_N(Y) = \left(\frac{Y}{Nx}\right)^{\frac{N-1}{2}} \exp\left[-Y-Nx\right] I_{N-1}\left(2\sqrt{NxY}\right) \tag{20}$$

The probability of detection is given by

$$P_D(N) = \int_c^{\infty} f_N(Y)\, dY \tag{21}$$

This integral is not soluable in terms of well known functions. It is a special case of the incomplete Toronto function [4] which is defined as

$$T_B(m,n,r) = 2r^{n-m+1} e^{-r^2} \int_0^B t^{m-n} e^{-t^2} I_n(2rt)\, dt \tag{22}$$

In terms of this function, the detection probability is

$$P_D(N) = 1 - T_{\sqrt{c}} \quad (2N-1,\ N-1,\ \sqrt{Nx}\ ) \qquad (23)$$

Curves of the incomplete Toronto function are available in reference 4 and numerical means of evaluating the function are available in the HMED programming library. For noise alone, an identical procedure may be carried out with the resultant equations:

Repeating Equation 13

$$f_o(y) = \exp\left[-y\right]$$

From Reference 3 pair 431 or direct integration

$$C_1(p) = \frac{1}{1+p} \qquad (24)$$

Then

$$C_N(p) = \frac{1}{(1+p)^N} \qquad (25)$$

From Reference 3 pair 431

$$f_o(Y) = \frac{Y^{N-1}\ \exp\left[-Y\right]}{(N-1)!} \qquad (26)$$

The probability of false alarm

$$P_A(N) = \int_C^{\infty} f_o(Y)\ dY = 1 - I\left(\frac{C}{\sqrt{N}}\ ,\ N-1\right) \qquad (27)$$

where I (.) is Pearsons form of the incomplete Gamma Function designed by the expression

$$I(u,p) = \int_0^{U\sqrt{p+1}} \frac{\exp\left[-v\right]\ V^P}{p!}\ dv \qquad (28)$$

The probability of false alarm as given by Equation 27 pertains to a particular possible target path. The number of such paths was defined in Section II as DA^{N-1}. For large A, the processor test functions, L(R,N,j), are approximately independent of each other and the probability that n of these result in a false alarm is

$$P_r \left\{ FA(N) = n \right\} = \binom{DA^{N-1}}{n} \left[P_A(N) \right]^n \left[1 - P_A(N) \right]^{DA^{N-1}-n} \tag{29}$$

The average number of false alarms is

$$\overline{FA(N)} = \sum_{n=1}^{DA^{N-1}} n \; P_r \left\{ FA(N) = n \right\} \tag{30}$$

The equation for $\overline{FA(N)}$ is seen to be equivalent to the expression for the mean of a binomial distribution which has the closed form result

$$\overline{FA(N)} = DA^{N-1} \; P_A(N) \tag{31}$$

With this definition of average number of false alarms, we note that DA^{N-1} appears as a simple multiplier. If we construct a plot of P_D -- or alternatively signal-to-noise ratio required for some fixed value of P_D -- versus $\overline{FA(N)}$, we need only plot against P_A with DA^{N-1} as a scale factor.

Equations 23 and 27 were evaluated by excercising existing in-house subroutines for the indicated functions for selected threshold values C and S/N ratio x. Curves of P_D versus P_A for fixed values of N and x, plotted for common values of C, are given in Figures 1 thru 6. A cross plot of this data is given in Figure 7 which shows the S/N ratio in db required to achieve a detection probability of 0.9 as a function of number of pings processed and probability of false alarm.

FIGURE I

Probability of Detection-Single Ping-Fixed S/N

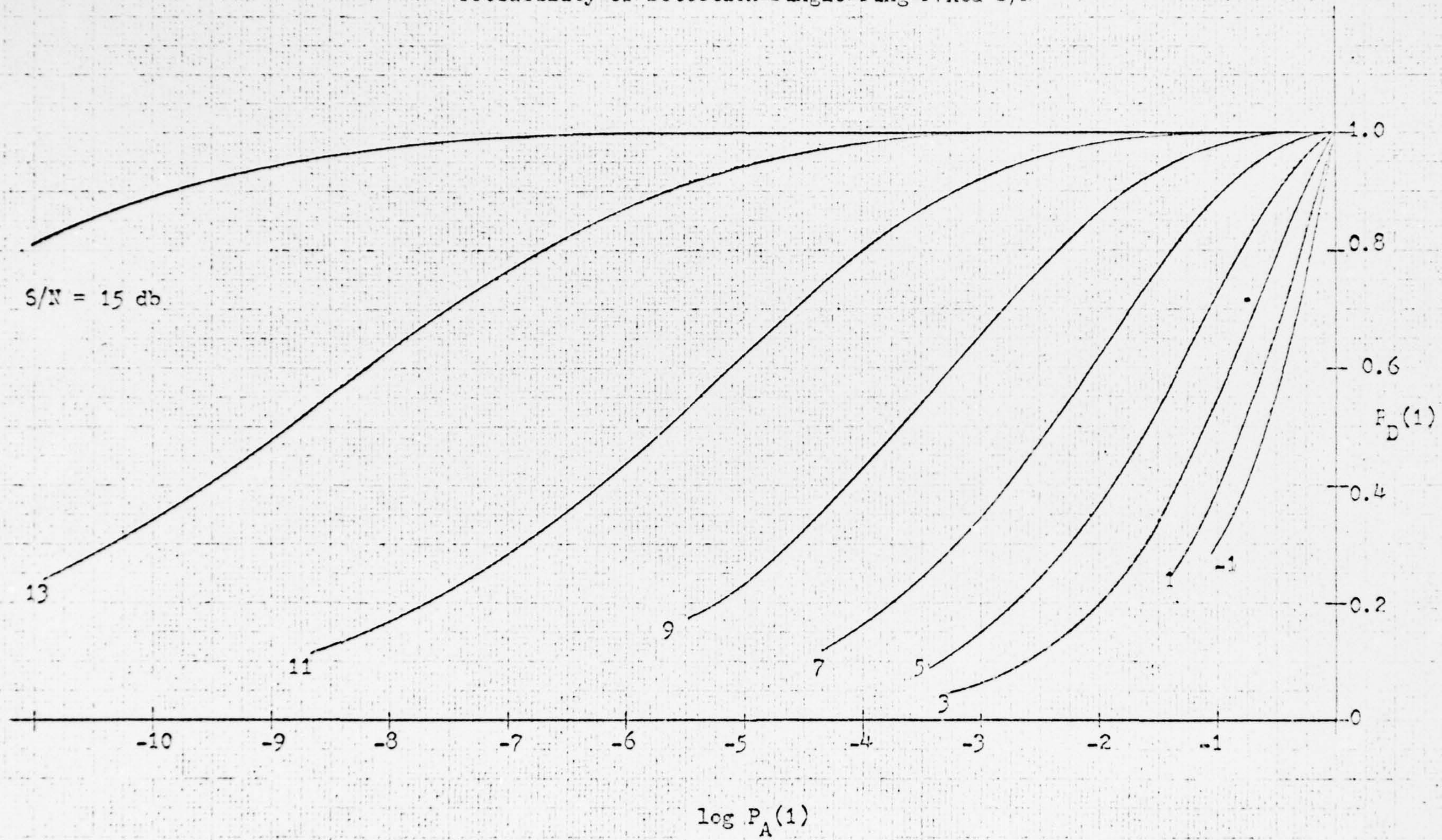

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FIGURE II

Probability of Detection-Two Ping Intervals-Fixed S/N

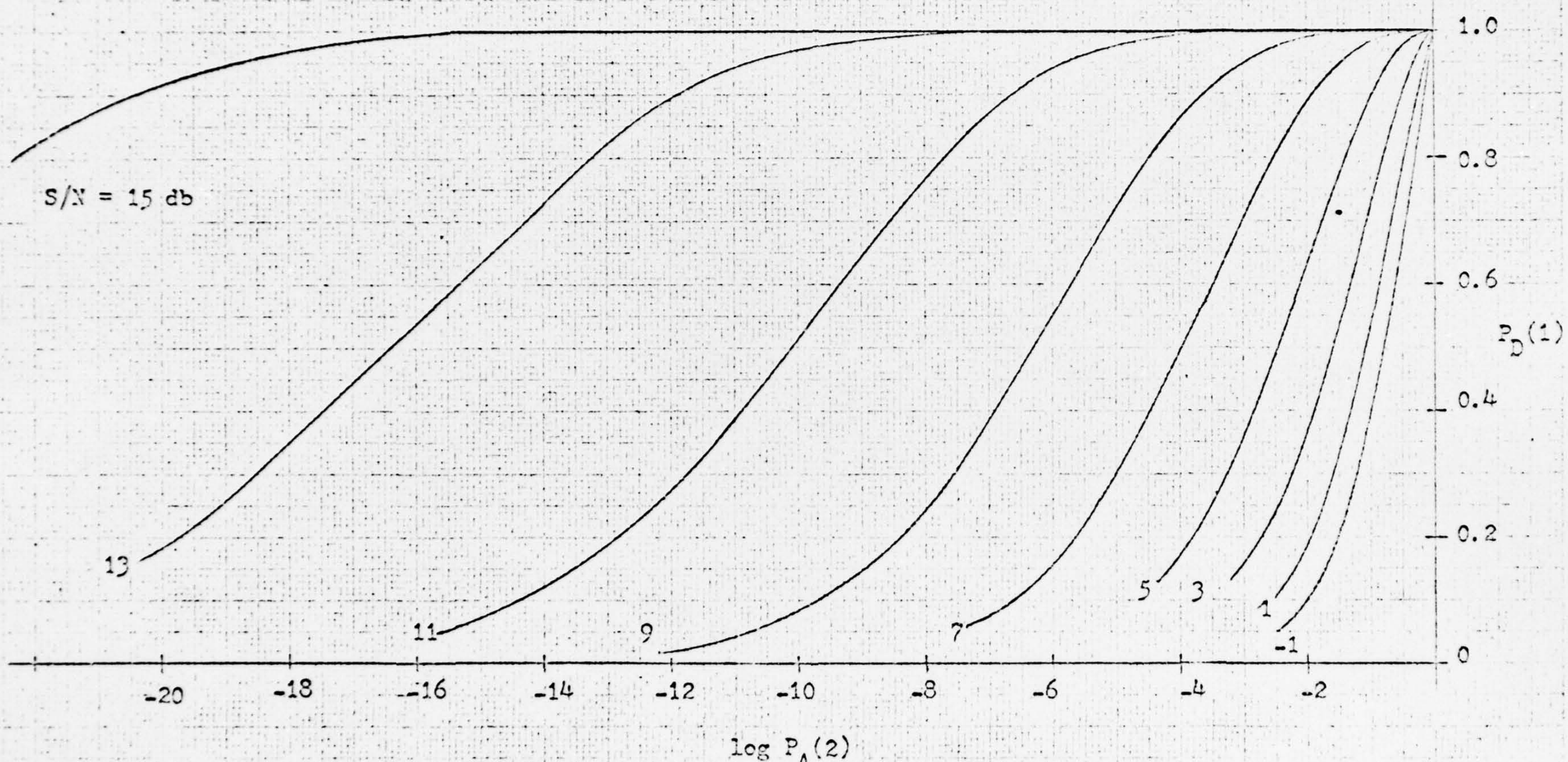

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FIGURE III

Probability of Detection-Three Ping Intervals-Fixed S/N

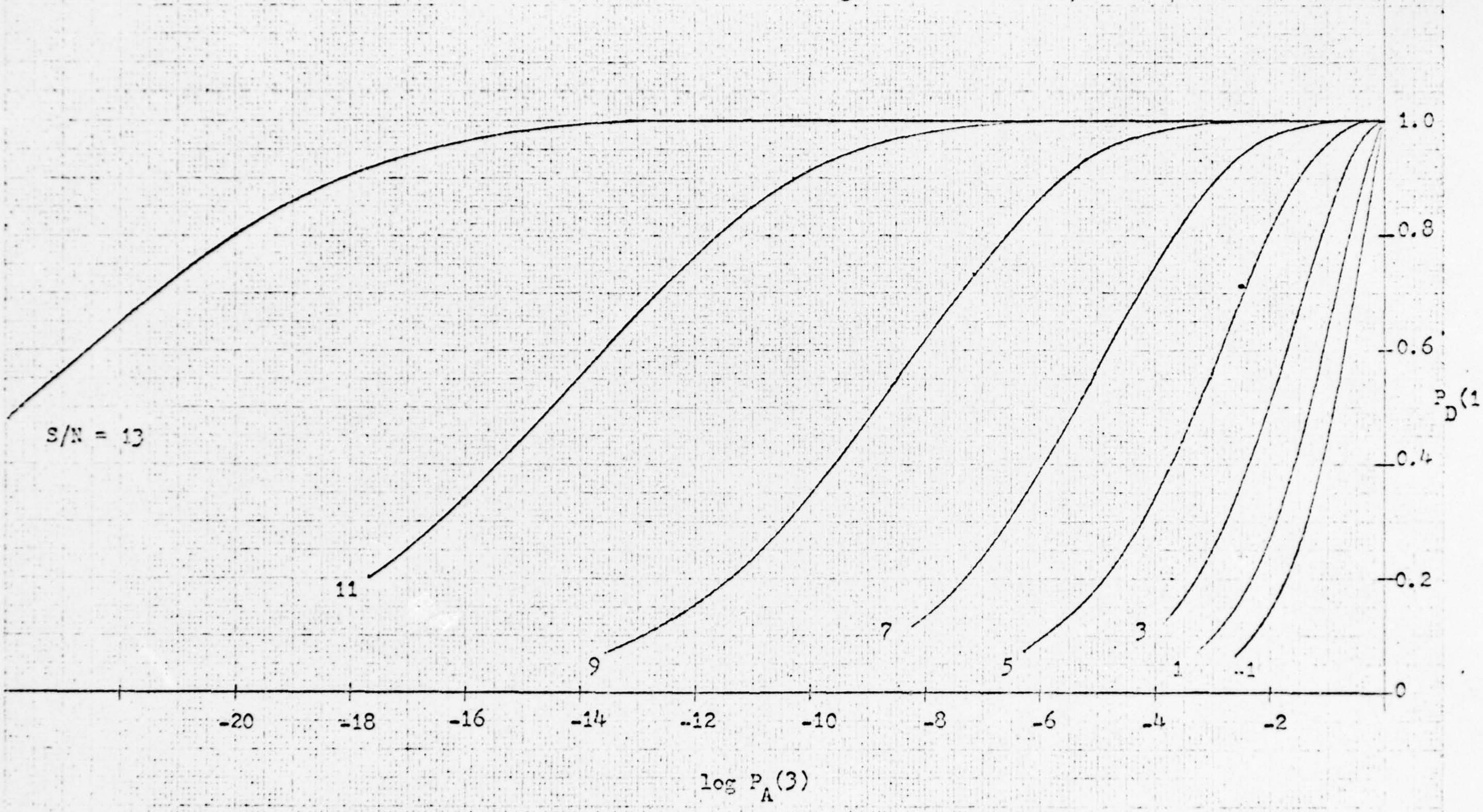

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FIGURE IV

Probability of Detection-Four Ping Intervals-Fixed S/N

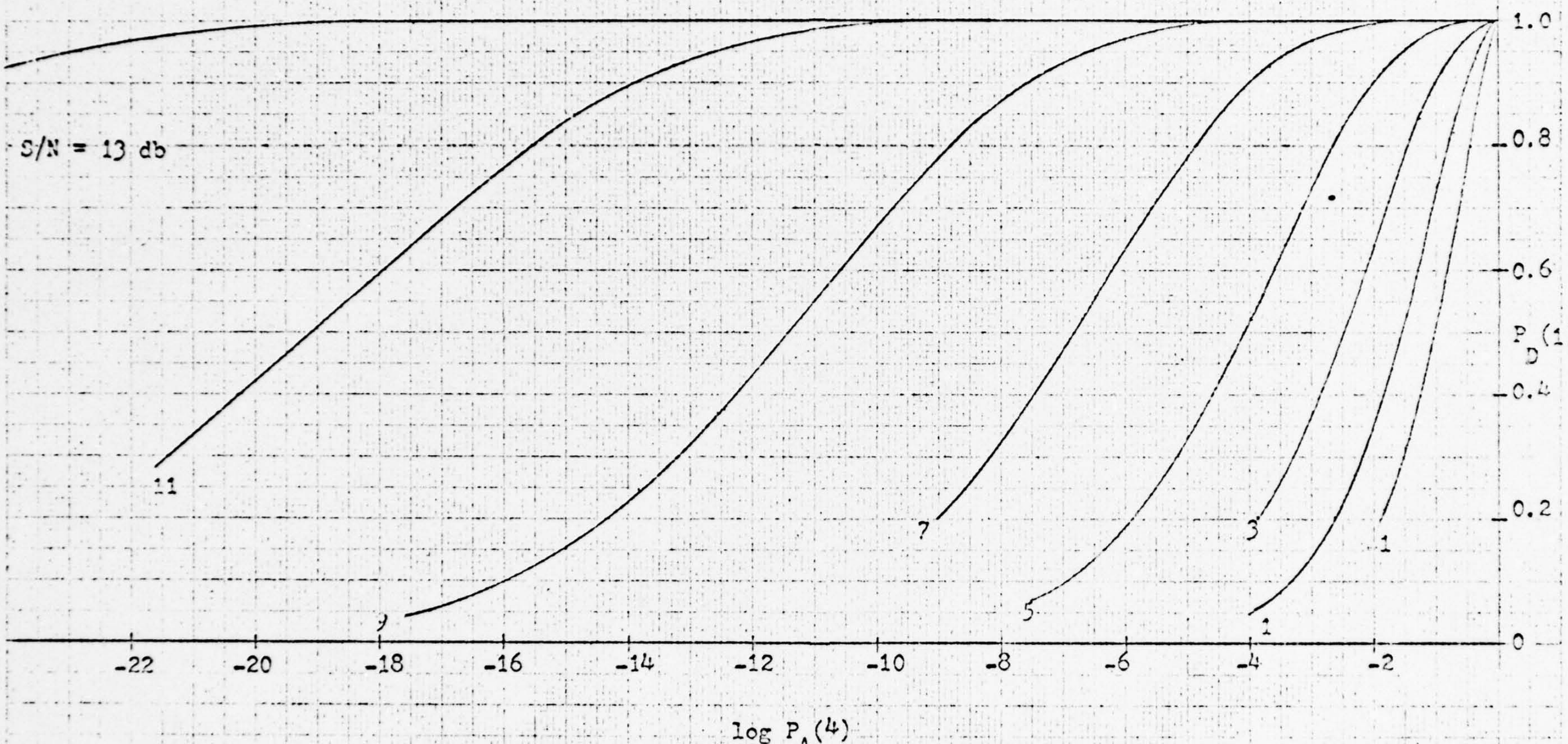

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FIGURE V

Probability of Detection-Five Ping Intervals-Fixed S/N

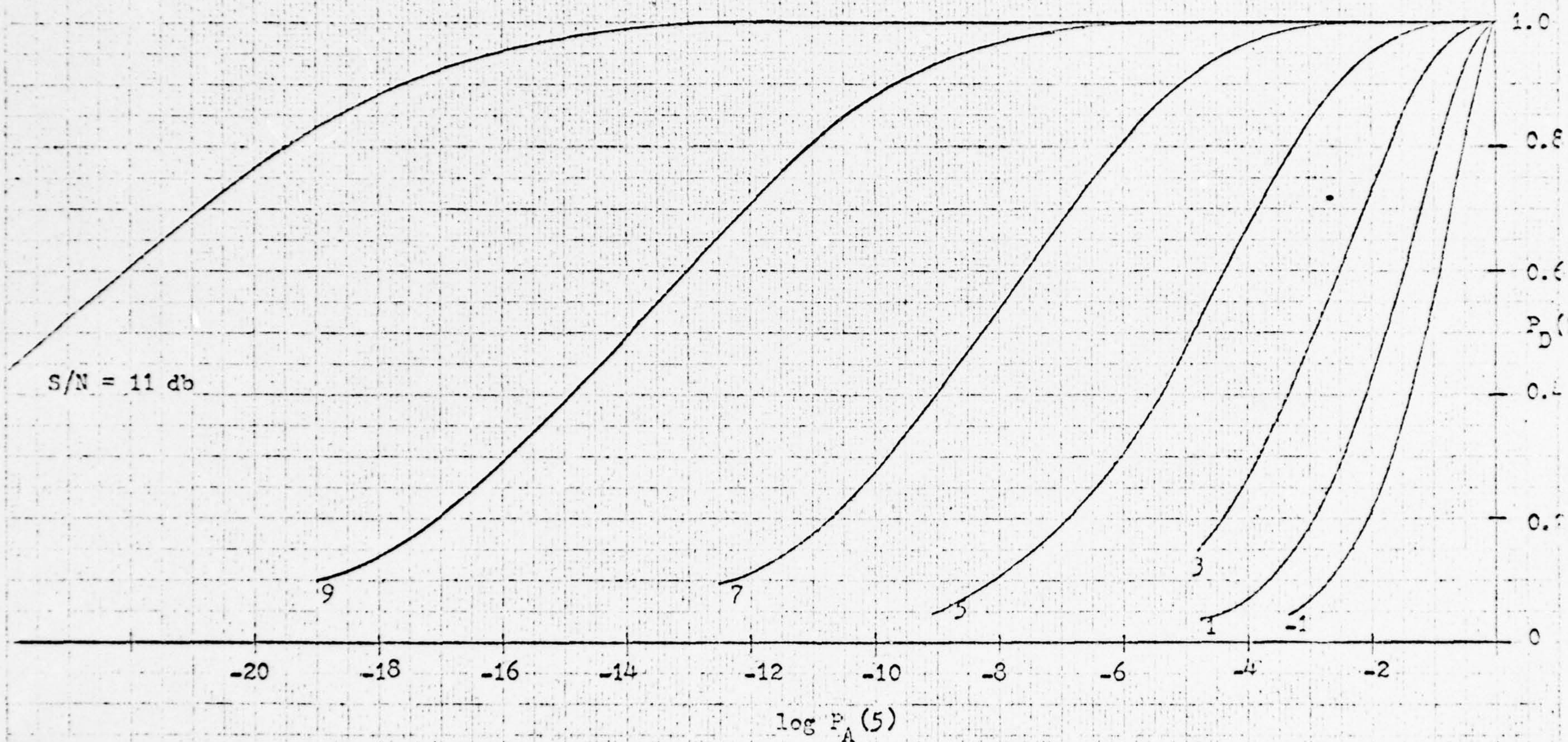

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FIGURE VI

Probability of Detection-Six Ping Intervals-Fixed S/N

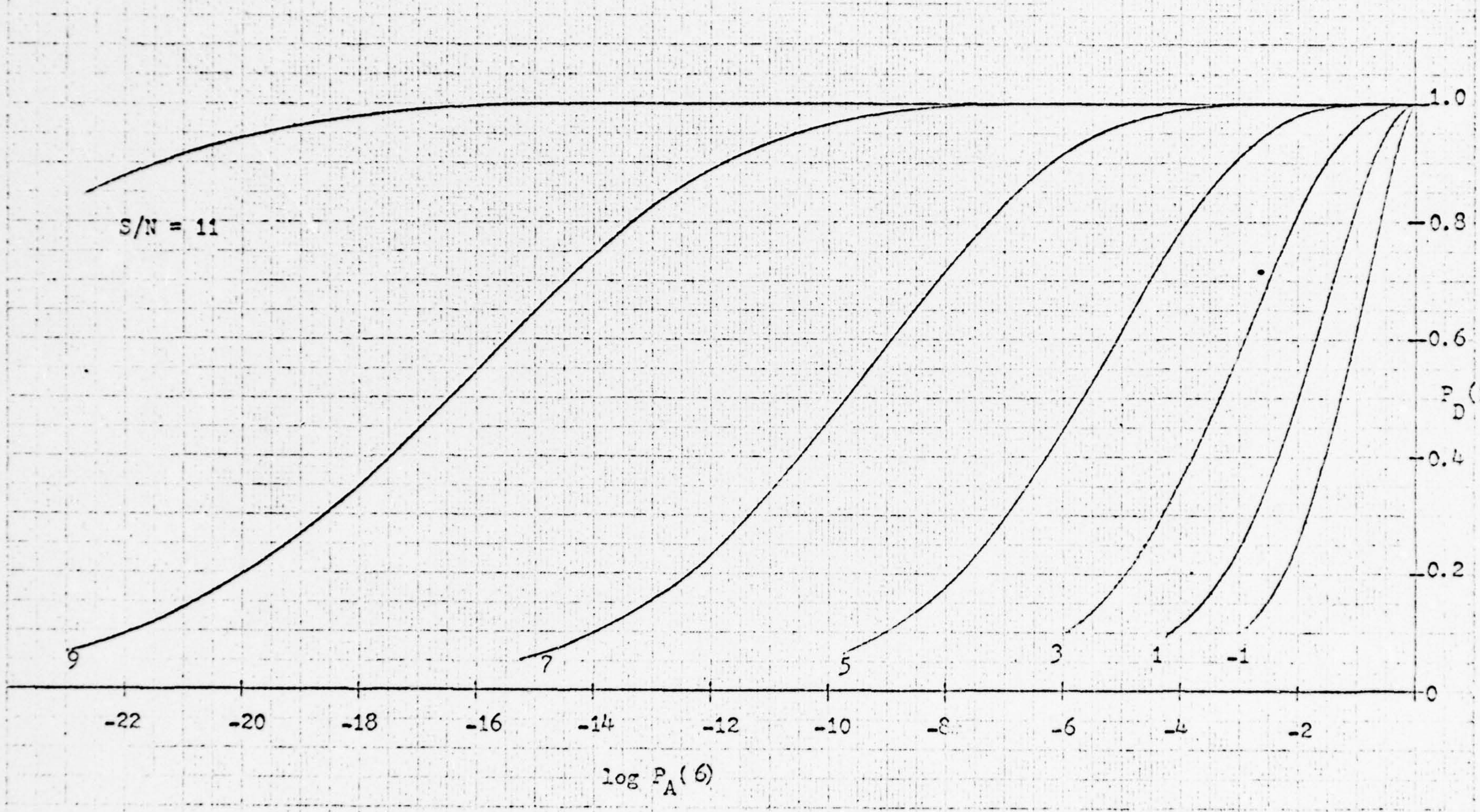

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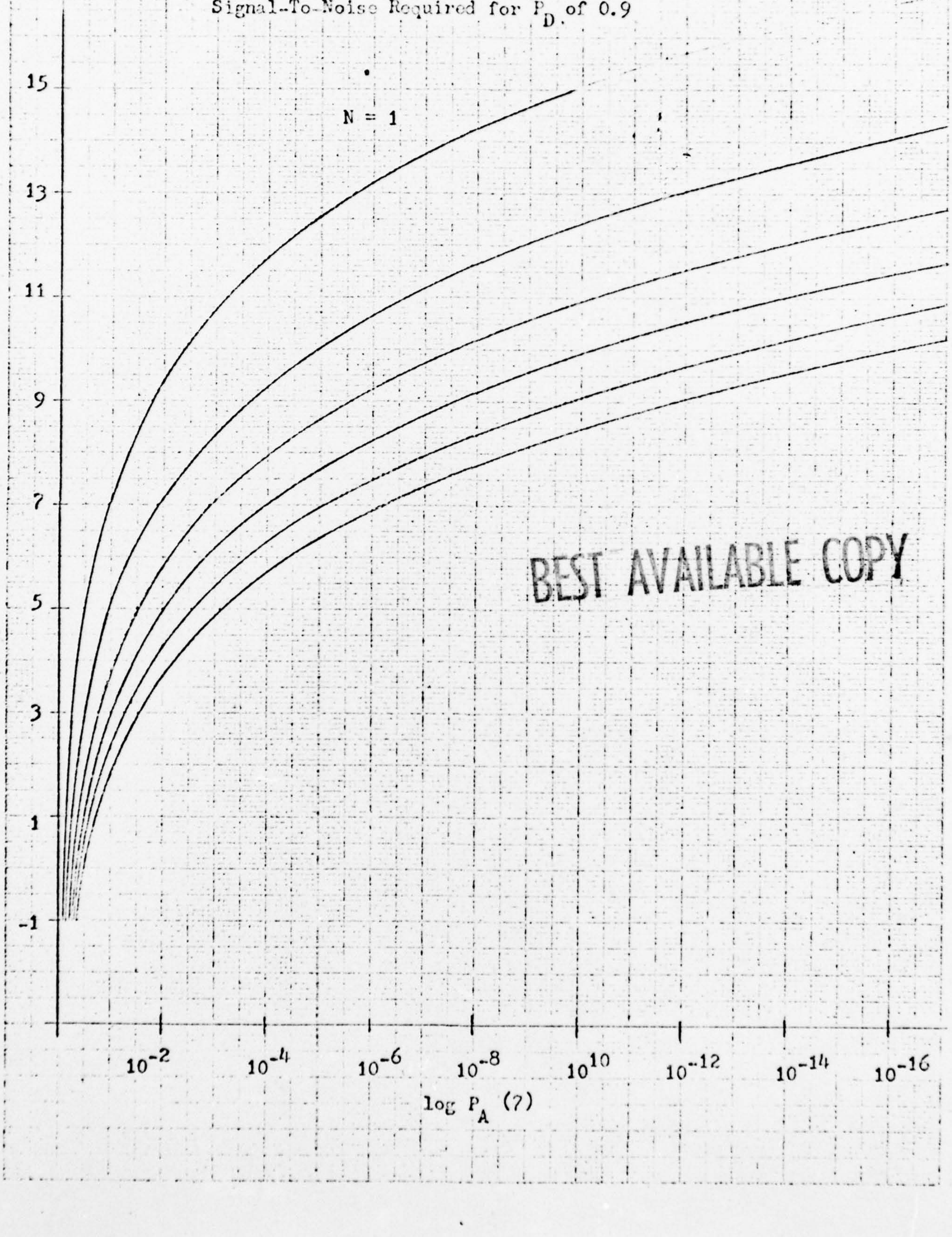
FIGURE VII

Signal-To-Noise Required for P_D of 0.9



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IV COMPARISON OF OPTIMUM AND M/N PROCESSOR--NON-SCINTILLATING TARGETS

A detailed treatment of an M/N processor is given in Reference 1. This processor operates under a target detection criteria in which M geometrically correlated single ping threshold crossings are required within a span of N ping intervals. A simpler treatment of that problem is given here to allow comparison of the M/N processor with the optimum processor.

The cumulative multi-ping probability of detection and false alarm are related to the single ping values by the equations:

$$P_D(N) = \sum_{k=M}^{N} \binom{N}{k} P_D(1)^k \left[1 - P_D(1)\right]^{N-k} \tag{32}$$

$$P_A(N) = \sum_{k=M}^{N} \binom{N}{k} P_A(1)^k \left[1 - P_A(1)\right]^{N-k} \tag{33}$$

As in the optimum processor, the average number of false alarms is given by

$$\widehat{FA(N)} = DA^{N-1} \; P_A(N) \tag{34}$$

In the comparison which follows, N is fixed at five ping intervals and the value of D and A apply equally well to both optimum and M/N processors. Thus, a full comparison of the two types of processors may be accomplished without considering the explicite values of D and A.

The single ping probability of detection and false alarm are related to each other by their common threshold. From Equation 15A, we write

$$P_A(1) = \int_c^{\infty} \exp\left[-y\right] = \exp\left[-c\right] \tag{35}$$

thus the threshold may be written directly in terms of false probability as

$$c = \ln\left(\frac{1}{P_A(1)}\right) \tag{36}$$

Applying this threshold to Equation 15-B yields for probability of detection:

$$P_D(1) = \int_{\ln\left(\frac{1}{P_A(1)}\right)}^{\infty} \exp\left[-y-x\right] \; I_o \; (2\sqrt{xy}) \; dy \tag{37}$$

This integral has no closed form and is generally expressed as Marcums Q function [4]

$$P_D(1) = Q\left(\sqrt{2x}\ ,\ \sqrt{2 \ln\ 1/P_A(1)}\right) \qquad (38)$$

where Q is defined by

$$Q(\alpha, \beta) = \int_{\beta}^{\infty} \exp\left[-\frac{v^2+\alpha^2}{2}\right] I_o(\alpha v)\ dv$$

The computational procedure for the comparison included here is as follows:

1) Set N equal to 5 and $P_D(5)$ equal to 0.9.
2) Using Equation 32 find $P_D(1)$ for M equal to 1 through 5.
3) For each M, use Equation 38 to find the S/N ratio required for selected values of $P_A(1)$
4) Using Equation 33 find the $P_A(5)$'s which correspond to the $P_A(1)$ of step 4.
5) Plot the S/N required versus $P_A(5)$ for each M

The results of these calculations together with the optimum processor for N equal to five are given in Figure 8. In general, the optimum shows about a two db advantage over the M/N processor.

FIGURE VIII

Comparison of Optimum versus M/N Processor

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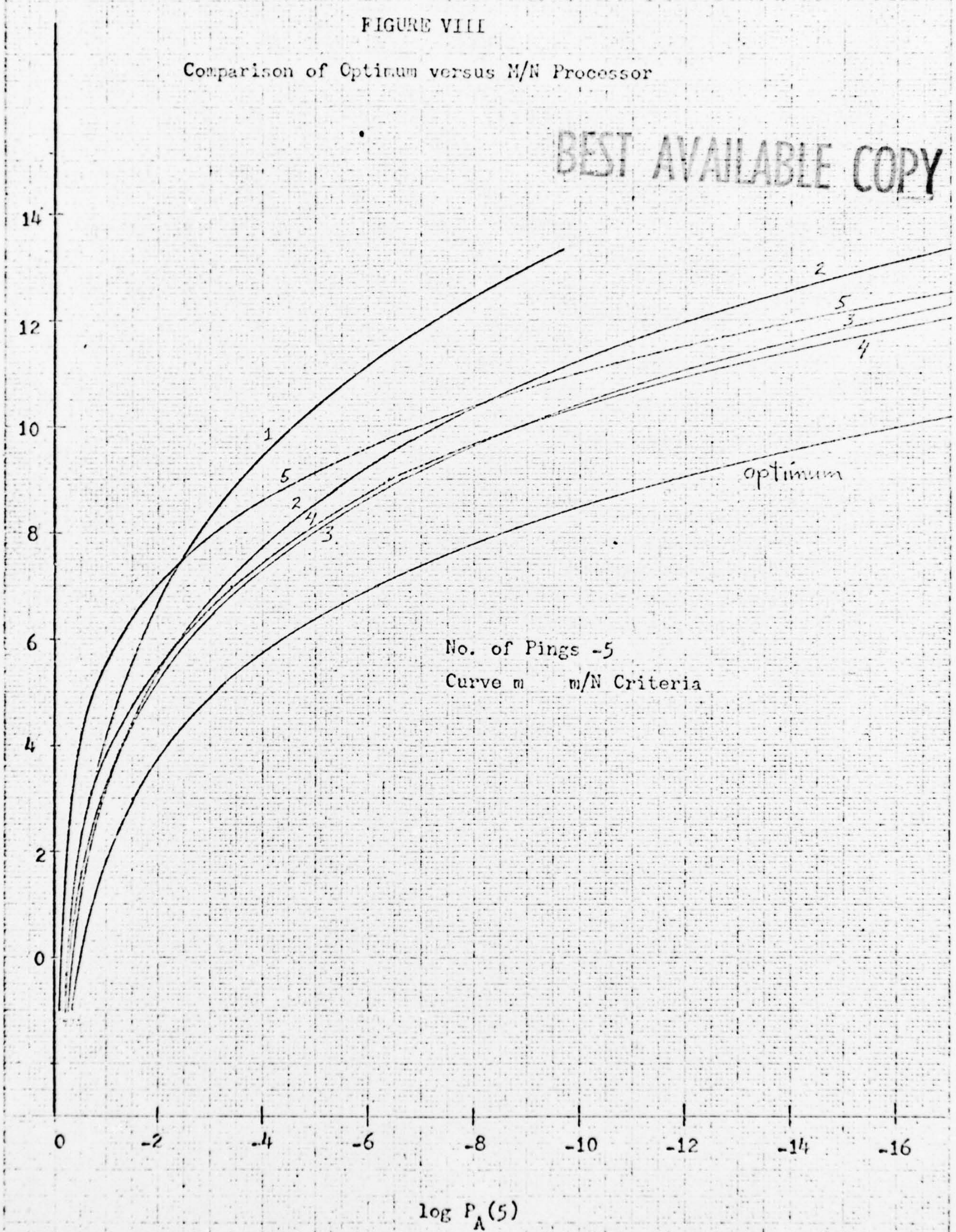


V SCINTILLATING TARGET MODELS

The analysis to this point utilizes a fixed S/N ratio at the input to the receiver. That this is not the case is well known and suitable modifications must be included in the analysis before it can be considered realistic.

The most accepted model representative of variation in signal-to-noise is the log-normal distribution. This model has been extensively used in signal ping analyses and does have justification from both theoretical and experimental standpoints.

Accepting the log-normal distribution as the best target scintillation model allows writting the distribution in the form:

$$f(Z) = \frac{1}{\sqrt{2\pi}\,\sigma_Z} \exp\left[-\frac{(Z - \overline{Z})^2}{2\sigma_Z^2}\right] \qquad (39)$$

where Z = S/N power ratio in db

$\overline{Z}$ = mean of Z

Z = standard deviation of Z

The general formulation of performance analysis requires expressing the distribution in terms of S/N power ratio in volts. The change of variable required is:

$$Z = 10 \log x = 10 \log e \;\; \ln x \qquad (40)$$

Completing the formal change of variable yields

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x \, x} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \overline{x}}{\sigma_x}\right)^2\right] \qquad (41)$$

where

$$\overline{x} = \frac{\overline{Z}}{10 \log e} \qquad \sigma_x = \frac{\sigma_Z}{10 \log e}$$

Attempts at forcing the distribution of Equation 41 into the analysis of Section III proved hopeless. The principle difficulty was the appearance of $(\ln x)^2$ terms which cannot be conveniently represented in series form.

To circumvent the difficulty, a distribution was assumed which had characteristics similar to the log normal and the analysis completed using the assumed distribution.

A family of scintillating target models is presented by Swerling [5] with the general equation.

$$w(x,\bar{x}) = \frac{1}{(K-1)!} \quad \frac{K}{\bar{x}} \quad \left(\frac{Kx}{\bar{x}}\right)^{K-1} \quad \exp\left[-\frac{Kx}{\bar{x}}\right] \tag{42}$$

The equation is a Chi square distribution with 2K degrees of freedom where x and $\bar{x}$ have the same interpretation as in Equation 41.

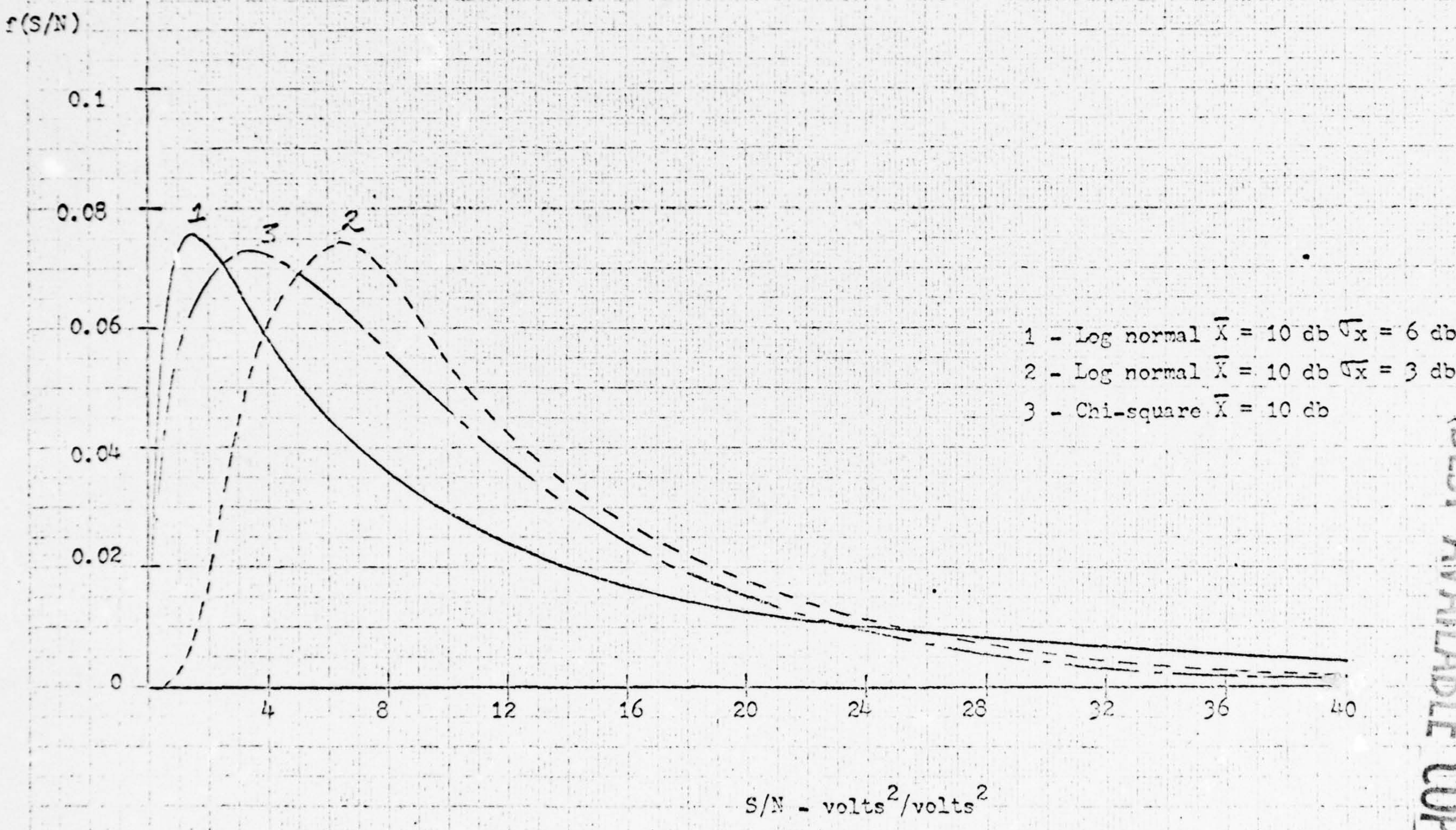
The best value of K proved to be 3/2 -- a Chi square distribution with 3 degrees of freedom. For this value of K, the equation for S/N distribution becomes

$$w(x,\bar{x}) = \frac{2}{\sqrt{\pi}} \quad \left(\frac{3}{2\,\bar{x}}\right)^{3/2} \quad x^{\frac{1}{2}} \quad \exp\left[-\frac{3x}{2\bar{x}}\right] \tag{43}$$

A comparitive plot of Equations 41 and 43 are shown in Figure 9. The use of Equation 43 as the S/N distribution appears reasonable particularly in view of the uncertainties inherent to Equation 41.

FIGURE IX

Density Distribution for Signal-To-Noise



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VI PERFORMANCE EVALUATION - SCINTILLATING TARGET

The signal plus noise distribution for fixed signal-to-noise ratio was given by Equation 14 as

$$f(y \mid x) = \exp\left[-y-x\right] \quad I_o \left(2\sqrt{xy}\right)$$

The assumed distribution for signal-to-noise variability was given by Equation 43 as

$$w(x) = \frac{2}{\sqrt{\pi}} \left(\frac{3}{2\bar{x}}\right)^{3/2} x^{\frac{1}{2}} \exp\left[-\frac{3x}{2\bar{x}}\right]$$

Given these distributions, the unconditional probability of y in general form is given by

$$f(y) = \int_x f(y \mid x)\ w(x)\ dx \tag{44}$$

For our problem, the resultant equation is

$$f(y) = b \exp(-y) \int_0^{\infty} x^{\frac{1}{2}} \exp(-ax)\ I_o \left(2\sqrt{xy}\right)\ dy \tag{45}$$

where

$$b = \frac{3}{\bar{x}} \left(\frac{3}{2\pi}\right)^{\frac{1}{2}}$$

$$a = 1 + \frac{3}{2\bar{x}}$$

The Bessel function may be written in series form as

$$I_o \left(2\sqrt{xy}\right) = \sum_{k=o}^{\infty} \frac{(xy)^k}{(k!)^2} \tag{46}$$

which allows rewritting the integral of Equation 45 as

$$\sum_{k=o}^{\infty} \frac{(y)^k}{(k!)^2} \int_0^{\infty} x^{k+\frac{1}{2}} \exp\left[-ax\right] dx \tag{47}$$

To evaluate the integral of Equation 47, we note the Gamma Function is defined by

$$\int_0^{\infty} t^{Z-1} e^{-t}\ dt = \Gamma(Z) \tag{48}$$

Making an appropriate change of variable, the integral of Equation 47 may be written in the closed form

$$\left(\frac{1}{a}\right)^{k+3/2} \Gamma(k + 3/2)$$

The unconditional distribution of y thus becomes

$$f(y) = b \exp(-y) \sum_{k=o}^{\infty} \frac{y^k}{(k!)^2} \frac{\Gamma(k + 3/2)}{a^{k + 3/2}} \tag{49}$$

Equation 49 pertains to the distribution of a single pulse. The distribution for the sum of N pulses is obtained in a manner analogous to that used in Section III.

To simplify notation let

$$A_k = \frac{b}{k!} \frac{\Gamma(k + 3/2)}{a^{k + 3/2}} \tag{50}$$

Equation 49 may now be written as

$$f(y) = \sum_{k=o}^{\infty} \frac{A_k}{k!} y^k \exp\left[-y\right] \tag{51}$$

From Campbell and Forster, pair 431, the characteristic function for one pulse is

$$C_1(p) = \sum_{k=o}^{\infty} \frac{A_k}{(p+1)^{k+1}} \tag{52}$$

For the sum of N pulses, we write

$$C_N(p) = \sum_{k_1=o}^{\infty} \sum_{k_2=o}^{\infty} \cdots \sum_{k_N=o}^{\infty} \frac{A_{k1} A_{k2} \cdots A_{kN}}{(p+1)^{k1 + k_2 \cdots k_N + N}} \tag{53}$$

Using pair 431 as an inverse transform yields the distribution of the sum as

$$f(Y) = \sum_{k_1=o}^{\infty} \sum_{k_2=o}^{\infty} \cdots \sum_{k_N=o}^{\infty} \frac{A_{k1} A_{k2} \cdots A_{kN}}{(k_1+k_2 \cdots k_N + N - 1)!} \exp\left[-Y\right] Y^{k_1+k_2 \cdots k_N+N-1} \tag{54}$$

The probability of detection is obtained by integrating Equation 54 from the threshold C to infinity. Taking the Y terms only we have

$$\int_{C}^{\infty} \frac{\exp\left[-Y\right] \quad Y^{r}}{r!} \cdot dy \tag{55}$$

where

$r = k_1 + k_2 \ldots + k_N + N - 1$

This integral has the same form as Pearson Incomplete Gamma Function which is defined by Equation 28. Symbolically the integral is represented by

$$1 - I\left(\frac{C}{\sqrt{r+1}}, r\right)$$

The final result is

$$P_D(N) = \sum_{k_1=o}^{\infty} \sum_{k_2=o}^{\infty} \cdots \sum_{k_N=o}^{\infty} \left(A_{k1} A_{k2} \ldots A_{kN} \right) \left[1 - I\left(\frac{C}{\sqrt{r+1}}, r\right)\right] \tag{56}$$

where

$r = k_1 + k_2 \ldots + k_N + N - 1$

$I(\cdot)$ = Pearson Incomplete Gamma Function

$$A_{ki} = \frac{b}{(k_i)!} \quad \frac{\Gamma(k_i + 3/2)}{a^{k_i + 3/2}}$$

$$b = \frac{3}{\bar{x}} \left(\frac{3}{2\pi}\right)^{\frac{1}{2}}$$

$$a = 1 + \frac{3}{2\bar{x}}$$

The false alarm computation is identical to that given in Section III.

A comparison of optimum and M/N processors may be made in a manner analogous to that of Section IV. The only change required is a change in the signal to noise distribution function given by Equation 51.

Making this change, Equation 37 of Section IV becomes

$$P_D(1) = \int_{\ln\left(\frac{1}{P_A(1)}\right)}^{\infty} \sum_{k=o}^{\infty} \frac{A_k}{k!} \quad y^k \quad \exp\left[-y\right] \quad dy$$

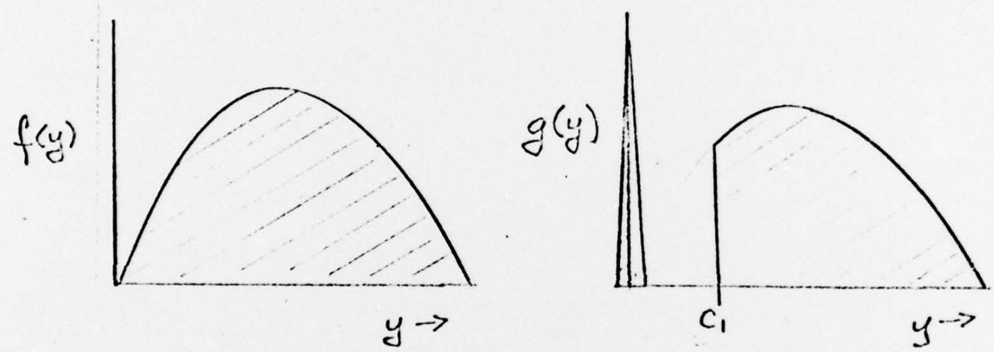
The integral is again recognizable as Pearsons Incomplete Gamma Function with the final result.

$$P_D(1) = \sum_{k=o}^{\infty} A_k \left\{ 1 - I \left(\frac{\ln \frac{1}{P_A(1)}}{\sqrt{k+1}} , k \right) \right\}$$

VII EFFECT OF THRESHOLDING

Use of an optimum processor implies the processing of enormous amounts of data. It is apparent that a practical processor must contain data reducing techniques to allow computational feasibility. One such technique is thresholding the single ping output prior to multi-ping processing. The multi-ping processor then concerns itself only with outputs which exceed a preset voltage level. The overall process becomes a double threshold device and performance is a function of the setting of the two thresholds.

The effect of single ping thresholding on the detector output is pictorially shown below



That is, thresholding places an impulse at y equal to zero, equal to the probability that y is less than C_1.

The distribution function may be written as

$$g(y) = P_r\left\{y < c_1\right\} \delta(y) + f(y) \quad u\,(y-C_1) \tag{57}$$

where

$\delta(y)$ = Dirac Delta Function

$$u(y-C_1) = \begin{cases} 0 & y \leq C_1 \\ 1 & y > C_1 \end{cases}$$

For noise alone, the distribution was previously shown to be given by

$$f(y) = \exp\left[-y\right]$$

For a fixed value of C_1, the value of $P_r \left\{ y < C_1 \right\}$ becomes a constant of the problem and is equal to

$$A_1 = \int_0^{C_1} \exp\left[-y\right] dy \tag{58}$$

$$= 1 - \exp\left[-C_1\right]$$

The threshold distribution function now becomes

$$g(y) = A_1 \, \delta(y) + \exp\left[-y\right] u\,(y-C_1) \tag{59}$$

The distribution for the sum of N pulses is again derived using characteristic functions. The Fourier transform of the delta function is unity. The transform of the remaining term is obtained by direct integration as follows

$$\int_0^{\infty} \exp\left[-y\right] u(y-C_1) \exp\left[-py\right] dy$$

$$= \int_c^{\infty} \exp\left[-(1+p)\,y\right] dy$$

$$= \frac{1}{1+p} \exp\left[-(1+p)\,C_1\right] \tag{60}$$

The characteristic function for a single pulse may be written as

$$C_1(p) = A_1 + \frac{1}{(1+p)} \exp\left[-(1+p)\,C_1\right] \tag{61}$$

The characteristic function for the sum of N pulses is the N-fold product of Equation 61. Using the binomial theorem, the resultant may be written in finite series form as

$$G_N(p) = \sum_{k=0}^{N} \binom{N}{k} A_1^{N-k} \frac{1}{(1+p)^k} \exp\left[-(1+p)\,k\,C_1\right] \tag{62}$$

The exponential in p is simply a displacement -- i.e. from Campbell and Forster, Pair 207

$$\exp\left[-pgo\right] \quad F \Rightarrow \quad G\,(g - g_o)$$

The inverse transform is again obtained from Campbell and Forster, pair 431 with the result.

$$g(Y) = \sum_{k=o}^{N} \binom{N}{k} A_1^{N-k} \frac{(Y-kC_1)^{k-1}}{(k-1)!} \exp\left[-Y\right] \qquad (63)$$

$$y > kC_1$$

The probability of false alarm is obtained from Equation 63 by integrating from the multi-ping threshold C_N to infinity. For the terms containing Y, the integral is

$$\int_{C_N}^{\infty} \frac{(Y - kC_1)^{k-1}}{(k-1)!} \exp\left[-Y\right] dY \qquad (64)$$

To perform the integration, first make the change of variable

$$Z = y - kc$$

which results in the expression

$$\exp\left[-kC_1\right] \int_{C_N-kC_1}^{\infty} \frac{Z^{k-1}}{(k-1)!} \exp\left[-Z\right] dz \qquad (65)$$

The latter integral has the form of Pearsons Incomplete Gamma Function defined by Equation 28 yielding for the integration

$$\exp\left[-kC_1\right] \left[1 - I\left(\frac{C_N-kC_1}{\sqrt{R}},\ k-1\right)\right]$$

The final expression for probability of false alarm is

$$P_A(N) = \sum_{k=o}^{N} \binom{N}{k} A_1^{N-k} \exp\left[-kC_1\right] \left[1 - I\left(\frac{C_N-kC_1}{\sqrt{k}},\ k-1\right)\right] \qquad (66)$$

$$C_N \doteq kC_1$$

where

$$A_1 = 1 - \exp\left[-C_1\right]$$

Computation of the average number of false alarms is identical to the earlier development in Section III and is given by Equation 31.

When a steady target is present, the distribution of y is given by

$$f(y) = \exp\left[-x-y\right] \; I_o \; (2\sqrt{xy})$$

The probability that y is less than the single ping threshold value is easily shown to be expressible in terms of Marcums Q function

$$P_r \left\{ f(y) < C_1 \right\} \equiv A_2 = 1 - Q(\sqrt{2x}, \sqrt{2C_1})$$

The thresholded distribution then has the form

$$g(y) = A_2 \, \delta(y) + \exp\left[-y-x\right] \; I_o \sqrt{2xy} \quad u(y-C_1) \qquad (67)$$

Using the series expansion for I_o given by Equation 46 yields

$$g(y) = A_2 \, \delta(y) + \sum_{k=o}^{\infty} \exp\left[-x-y\right] \frac{(xy)^k}{(k!)^2} \quad u(y-C_1) \qquad (68)$$

A general development for computing the probability of detection for the sum of N pulses given a single ping distribution in the form of Equation 68 is given in Appendix A. Direct application of that work yields the result.

$$P_D(N) = \sum_{m=o}^{N} \binom{N}{m} A_2^{N-m} \exp\left[-m(x+C_1)\right] \cdot$$

$$\sum_{k_1=o}^{\infty} \cdots \sum_{k_m=0}^{\infty} \frac{x^{k_1 + \dots k_m}}{(k_1)! \dots (k_m)!} \cdot \qquad (69)$$

$$\sum_{l_1=o}^{k_1} \cdots \sum_{l_m=o}^{k_m} \frac{C_1^{(k_1-l_1) + \dots (k_m-l_m)}}{(k_1-l_1)! \dots (k_m-l_m)!} \left[1 - I\left(\frac{C_N - mC_1}{\sqrt{r}}, r-1\right)\right]$$

$$C_N \geq mC_1$$

$$r = l_1 + \dots l_m + m$$

$$A_2 = 1 - Q\,(\sqrt{2x}, \sqrt{2C_1})$$

For a scintillating target, the distribution as given by Equation 51 is

$$f(y) = \sum_{k=o}^{\infty} \frac{A_k}{k!} y^k \exp\left[-y\right]$$

The probability that f(y) is less than the single ping threshold C_1 is

$$P_r\left\{y \leq C_1\right\} = A_2 = \sum_{k=o}^{\infty} A_k \int_o^{C_1} \frac{y^k \exp\left[-y\right]}{k!} dy$$

$$= \sum_{k=o}^{\infty} A_k \, I\left(\frac{C_1}{\sqrt{k+1}}, k\right) \tag{70}$$

where I(·) is the Incomplete Gamma Function of Equation 28. The thresholded function thus becomes

$$g(y) = A_2 \, \delta(y) + \sum_{k=o}^{\infty} \frac{A_k}{k!} y^k \exp\left[-y\right] u(y-C_1) \tag{71}$$

Applying the general development of Appendix A to Equation 71 yields

$$P_D(N) = \sum_{m=o}^{N} \binom{N}{m} A_2^{N-m} \exp\left[-mC_1\right] \sum_{k_1=o}^{\infty} \cdots \sum_{k_m=o}^{\infty} A_{k1} \ldots A_{km} \cdot$$

$$\sum_{l_1=o}^{k_1} \cdots \sum_{l_m=o}^{k_m} \frac{C^{(k_1-l_1) + \ldots (k_m-l_m)}}{(k_1-l_1)! \ldots (k_m-l_m)!} \left[1 - I\left(\frac{C_N-mC_1}{\sqrt{r}}, r-1\right)\right]$$

$$C_N \geq mC_1$$

where

$$r = l_1 + \ldots l_m + m$$

$$A_1 = \sum_{k=o}^{\infty} A_k \, I\left(\frac{C_1}{\sqrt{k+1}}, k\right)$$

$$A_k = \frac{b}{k!} \frac{\Gamma(k+3/2)}{a^{k+3/2}}$$

$$b = \frac{3}{\bar{x}} \left(\frac{3}{2\pi}\right)^{\frac{1}{2}} .$$

$$a = 1 + \frac{3}{2\bar{x}}$$

APPENDIX A - THRESHOLDED MULTI-PING PERFORMANCE

For several assumed signal-plus-noise distributions, the expanded form of the distribution has the general equation

$$g(y) = A_1 \delta(y) + \sum_{k=0}^{\infty} A_k y^k \exp\left[-y\right] u\ (y-c) \qquad (A-1)$$

where y = normalized output of a square law detector

$\delta(y)$ = unit impulse at $y = 0$

$u(y-c)$ = unit step at $y = c$

The probability of detection for the sum of N pulses from the above distribution function is desired. This is obtained by the usual application of characteristic functions. In what follows, the characteristic function is taken as the Fourier transform allowing use of Campbell and Forster Fourier Transform Tables.

The Fourier Transform of the impulse function is unity. The transform of the second term of Equation A-1 procedes as follows:

Considering the y terms within the summation, the transform in integral form is

$$\int_{-\infty}^{\infty} y^k \exp\left[-y\right] u\ (y-c) \exp\left[-py\right] dy \qquad (A-2)$$

The unit step at $y = c$ may be eliminated by setting the lower limit at C

$$\int_{C}^{\infty} y^k \exp\left[-y\right] \exp\left[-py\right] dy \qquad (A-3)$$

Since Fourier transforms are available with lower limit equal to zero, we now make the change of variable

$$Z = y-c$$

$$dZ = dy$$

which yields the equivalent expression

$$\int_{0}^{\infty} (Z+C)^k \exp\left[-(Z+C)\right] \exp\left[-p(Z+C)\right] dZ \qquad (A-4)$$

The binomial term may now be written as a finite series with the result

$$\int_{0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} c^{k-l} Z^l \exp\left[-Z\right] \exp\left[-(1+p)\ C\right] \exp\left[-pZ\right] dZ \qquad (A-5)$$

Campbell and Forster, pair 431, present the transform

$$\frac{1}{(k-1)!} g^{k-1} e^{-\beta g} \Rightarrow \frac{1}{(p+\beta)^k} \qquad g > 0 \tag{A-6}$$

Taking the integral inside the summation and applying pair 431 yields

$$z^l \exp\left[-z\right] \Rightarrow \frac{l!}{(p+1)^{l+1}}$$

The Fourier transform of Equation A-1 may now be written as

$$C_1(p) = A_1 + \sum_{k=0}^{\infty} A_k \sum_{l=0}^{k} \frac{k! \; C^{(k-l)} \exp\left[-(1+p)C\right]}{(k-l)! \; (p+1)^{l+1}} \tag{A-7}$$

The Fourier transform for the sum of N pulses is the N-fold multiplication of Equation A-7. Symbolically treating the right hand side of Equation A-7 as the sum of two terms, we write

$$C_N(p) = \left[C_1(p)\right]^N = (a+b)^N$$

$$= \sum_{m=0}^{N} \binom{N}{m} a^{N-m} b^m$$

Inserting the appropriate expressions for a and b yields

$$C_N(p) = \sum_{m=0}^{N} \binom{N}{m} A_1^{N-m} \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} A_{k1} \cdots A_{km}$$

$$\sum_{l_1=0}^{k_1} \cdots \sum_{l_m=0}^{k_m} \frac{k_1! \ldots k_m! \exp\left[-m(1+p)C\right] C^{(k_1-l_1)+\ldots(k_m-l_m)}}{(k_1-l_1)! \ldots (k_m-l_m)! \; (p+1)^{l_1+\ldots l_m+m}} \tag{A-8}$$

The next step is to take the inverse transform. The only terms we need consider are those containing p. These are:

$$\frac{\exp\left[-pmc\right]}{(p+1)^r} \qquad r = l_1 + \ldots . l_m + m \tag{A-9}$$

The exponential term represents a displacement, i.e.

$$C(p)\, e^{-pYo} \Rightarrow g(Y\text{-}Yo)$$

The denominator of Equation A-9 has the same form as the transform pair given in Equation A-6. The resultant transform of Equation A-9 is

$$\frac{\exp\left[-pmc\right]}{(p+1)^r} \Rightarrow \frac{1}{(r-1)!} \quad (Y-mc)^{r-1} \quad \exp\left[(-Y-mc)\right] \qquad \text{(A-10)}$$

The integral of g(Y) from a multi-ping threshold, C_N, to infinity yields the probability of detection. To carry out the integration we may restrict ourselves to those terms containing Y as given by Equation A-10. The desired integral is

$$\exp\left[mc\right] \int_{C_N}^{\infty} \frac{(Y-mc)^{r-1} \exp\left[-Y\right]}{(r-1)!} \, dY \qquad \text{(A-11)}$$

To carry out the integration, first make the change of variable

$$Z = y - mc$$
$$dZ = dy$$

which yields the equivalent integral

$$\int_{C_N - mc}^{\infty} \frac{Z^{r-1} \exp\left[-Z\right]}{(r-1)!} \, dZ \qquad \text{(A-12)}$$

The latter integral may be identified with Pearson's Incomplete Gamma Function which is defined as

$$I\ (u,p) = \int_{o}^{u\sqrt{p+1}} \frac{t^p \exp\left[-p\right]}{p!} \, dt$$

In terms of this function, Equation A-12 becomes

$$1 - I\ \left(\frac{C_N - mc}{\sqrt{r}}\ ,\ r-1\right)$$

The final expression for probability of detection is

$$P_D(N) = \sum_{m=0}^{N} \binom{N}{m} A_1^{N-m} \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} A_{k1} \cdots A_{km} \exp\left[-mc\right] \cdot$$

$$\sum_{l_1=0}^{k_1} \cdots \sum_{l_m=0}^{k_m} \frac{k_1! \cdots k_m! \quad c^{(k_1-l_1)+\cdots(k_m-l_m)}}{(k_1-l_1)! \cdots (k_m-l_m)!} \cdot$$

$$\left\{ 1 - I\left(\frac{C_N - mc}{\sqrt{r}},\ r-1 \right) \right\}$$

$$C_N > mc$$

$$r = l_1 + \cdots l_m + m$$

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